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The intersection numbers of the p -spin curves from random matrix theory

E. Brézin^{a)} and S. Hikami^{b)}

^{a)} Laboratoire de Physique Théorique, Ecole Normale Supérieure
24 rue Lhomond 75231, Paris Cedex 05, France. e-mail: brezin@lpt.ens.fr¹

^{b)} Mathematical and Theoretical Physics Unit, OIST Graduate University,
1919-1 Tancha, Onna-son, Okinawa 904-0495 Japan. e-mail: hikami@oist.jp

Abstract

The intersection numbers of p -spin curves are computed through correlation functions of Gaussian ensembles of random matrices in an external matrix source. The p -dependence of intersection numbers is determined as polynomial in p ; the large p behavior is also considered. The analytic continuation of intersection numbers to negative values of p is discussed in relation to $SL(2, \mathbb{R})/U(1)$ black hole sigma model.

¹*Unité Mixte de Recherche 8549 du Centre National de la Recherche Scientifique et de l'École Normale Supérieure.*

1 Introduction

The intersection numbers for p -spin curves appear in the generalized Kontsevich matrix model [1, 2, 3]. The generating function for p -spin intersection number obeys the p -th KdV equation or Gelfand-Dikii equation. In a random matrix theory, the correlation functions at the edges of the spectrum, where one can tune a degeneracy of order p , are expressed through intersection numbers [4, 5, 6, 7]. In conformal field theory, the p -spin curve intersection theory is related to $\mathcal{N}=2$ superconformal minimal theory for Lie algebra A_{p-1} type. It has been pointed out that it corresponds to a gauged Wess-Zumino-Witten (WZW) model of $SU(2)_k/U(1)$, where $k = p - 2$ is the level of the Kac-Moody algebra of Lie group $SU(2)$ [8], and is related to $SL(2, \mathbb{R})/U(1)$ black hole sigma model when k becomes negative [9].

The free energy for the p -spin curve satisfies interesting universal equations, such as a string equation, dilaton equation, and WDVV equation, so called tautological equations or universal equations, and has been studied in the connection to Gromov-Witten theory [10, 11, 12]. Although the p -spin curve intersection numbers can be obtained through tautological equations in a recursive way, the actual computation for higher genres is limited [13, 14, 15].

In previous papers, we have derived explicit integral formula for the p -spin curve intersection numbers of the moduli space $\overline{M}_{g,n}$ valid for all order of genus g . We have shown that they are obtained analytically for a fixed number n of marked points. Our formulation starts from simple Gaussian matrix models with an external matrix source and based upon a duality relation [5, 6, 7], from which one recovers a generalized Kontsevich matrix model.

The intersection numbers for the spin moduli spaces with n -marked points are obtained from the n -point correlation functions $U(s_1, \dots, s_n)$ of Gaussian random matrices in a scaling limit at critical edges [16, 17]. In a previous article [18], we have computed explicitly the intersection numbers of moduli space of p -spin curves with one marked point, for arbitrary values of p , as polynomials in p . This allowed us to consider continuations in p ; in particular the limit $p \rightarrow -1$ exhibits an interesting relation between the intersection numbers, and the orbifold Euler characteristics $\chi(\overline{M}_{g,1}) = \zeta(1 - 2g)$, where $\zeta(x)$ is the Riemann zeta function) [20, 21].

In this paper, we extend the evaluation of the intersection numbers beyond the one marked point ($n = 1$) for arbitrary p . The obtained intersection numbers are consistent with previously known results [13, 14, 15] for small values of p . We pursue the large p behavior, $p \rightarrow \infty$ limit. The p -spin curve intersection theory is equivalent to gauged WZW model. For this gauged WZW theory, in which $k = p - 2$ appears as overall factor, the large k limit

may give a semi-classical solution [9, 23]. In the negative k , the gauged WZW model on $SU(2)/U(1)$ is changed to WZW model on non-compact $SL(2, R)/U(1)$, which is relevant to a black hole σ model [9]. We will discuss the relation between the intersection numbers and the density of state of $SL(2, R)/U(1)$ black hole sigma model [24, 25, 27].

2 Generating function for p -spin Intersection numbers

The mathematical definition of the intersection numbers of the moduli space of p -spin curves with s -marked points is given by [3]

$$\langle \tau_{n_1}(U_{j_1}) \cdots \tau_{n_s}(U_{j_s}) \rangle = \frac{1}{p^g} \int_{\overline{M}_{g,s}} C_T(\nu) \prod_{i=1}^s (c_1(\mathcal{L}_i))^{n_i} \quad (1)$$

where U_j is an operator for the primary matter field (tachyon), related to top Chern class $C_T(\nu)$, and τ_n is a gravitational operator, related to the first Chern class c_1 of the line bundle \mathcal{L}_i at the i th-marked point. We denote $\tau_n(U_j)$ by $\tau_{n,j}$, and j represents the spin index ($j=0, \dots, p-1$). The problem of definition (1) has been discussed extensively [11].

In a previous paper [18], we have shown that those intersection numbers (1) are expressed through the correlation functions $U(s_1, \dots, s_n)$ as coefficients of powers of s_j ,

$$\begin{aligned} U(s_1, s_2, \dots, s_n) &= \langle \text{tr} e^{s_1 M} \text{tr} e^{s_2 M} \cdots \text{tr} e^{s_n M} \rangle \\ &= \int \prod_{l=1}^n d\lambda_l e^{\sum_{l=1}^n i t_l \lambda_l} \langle \prod_{l=1}^n \text{tr} \delta(\lambda_l - M) \rangle \end{aligned} \quad (2)$$

where $s_l = i t_l$; M is an $N \times N$ Hermitian random matrix. The bracket stands for averages with the Gaussian probability measure

$$\langle X \rangle = \frac{1}{Z} \int dM e^{-\frac{N}{2} \text{tr} M^2 + N \text{tr} M A} X(M), \quad (3)$$

in which A is an $N \times N$ external Hermitian matrix source. By an appropriate tuning of the external source matrix A , we may obtain the desired singularity, which generates the p -spin curves. The relation to the generalized Kontsevich model is discussed in § 3,4 of [18].

An exact and useful integral representation for $U(s_1, \dots, s_n)$ is known in presence of an arbitrary external matrix source A with eigenvalues a_α [19] :

$$\begin{aligned} U(s_1, \dots, s_n) &= \frac{1}{N} \langle \text{tr} e^{s_1 M} \cdots \text{tr} e^{s_n M} \rangle \\ &= e^{\sum_{i=1}^n s_i^2} \oint \prod_{i=1}^n \frac{du_i}{2\pi i} e^{\sum_{i=1}^n u_i s_i} \prod_{\alpha=1}^N \prod_{i=1}^n \left(1 - \frac{s_i}{a_\alpha - u_i}\right) \det \frac{1}{u_i - u_j + s_i} \end{aligned} \quad (4)$$

This representation involves contour integrals around $u_i = a_\alpha$. In the large N limit, it is convenient to express the factors in the determinant as additional integrals. For instance, in the case of the two point correlation (n=2), after the shift $u_i \rightarrow u_i - \frac{s_i}{2}$, $s_i \rightarrow \frac{s_i}{N}$, in the two point function, we have

$$\begin{aligned} & \frac{1}{u_1 - u_2 + \frac{1}{2N}(s_1 + s_2)} \frac{1}{u_1 - u_2 - \frac{1}{2N}(s_1 + s_2)} \\ &= \frac{N}{s_1 + s_2} \int_0^\infty dx e^{-x(u_1 - u_2)} \text{sh}\left(\frac{x}{2N}(s_1 + s_2)\right) \end{aligned} \quad (5)$$

Tuning now the a_α 's, and taking the large N limit, we obtain

$$\begin{aligned} U(s_1, s_2) &= \frac{2N}{s_1 + s_2} \frac{1}{(2\pi i)^2} \int_0^\infty dx \int du_1 du_2 \text{sh}\left(\frac{1}{2N}x(s_1 + s_2)\right) e^{-(u_1 - u_2)x} \\ &\times \exp\left[-\frac{N}{p^2 - 1} \sum \frac{1}{d_\alpha^{p+1}} \left(\sum_i \left(u_i + \frac{1}{2N}s_i\right)^{p+1} - \sum_i \left(u_i - \frac{1}{2N}s_i\right)^{p+1}\right)\right] \end{aligned} \quad (6)$$

For the three and four point correlations, similar useful formulae for the determinant part of (4) may be found in the appendices A and B of [18].

3 Intersection numbers for $p = 3$ with two marked points

The intersection numbers are obtained as coefficients of the power series in s_1, s_2 of $U(s_1, s_2)$. In a previous paper [18], for p=3, we have computed the intersection numbers with two marked points or genus one case (g=1) starting from (7). As an example, we compute the $p = 3$ case up to genus 3. The general expansion

$$U(s_1, s_2) = \sum_{g, m, j} \langle \tau_{m_1, j_1} \tau_{m_2, j_2} \rangle_g \Gamma\left(1 - \frac{1 + j_1}{p}\right) \Gamma\left(1 - \frac{1 + j_2}{p}\right) s_1^{m'_1} s_2^{m'_2} \quad (7)$$

with the condition,

$$(p + 1)(2g - 2 + n) = \sum_{i=1}^s (pm_i + j_i + 1), \quad m'_k = m_k + \frac{1 + j_k}{p} \quad (k = 1, 2) \quad (8)$$

is applied to the special case $n = 2, p = 3$. The gamma functions in (7) represent the spin factors.

After rescaling of the parameters,

$$\begin{aligned} U(s_1, s_2) &= \frac{2}{(s_1 + s_2)(3s_2)^{1/3}} \int_0^\infty dy \text{sh}\left(\frac{s_1 + s_2}{2}(3s_1)^{1/3}y\right) A_i\left(y - \frac{1}{4 \cdot 3^{1/3}}s_1^{8/3}\right) \\ &\times A_i\left(-ay - \frac{1}{4 \cdot 3^{1/3}}s_2^{8/3}\right) \end{aligned} \quad (9)$$

in which $a = (s_1/s_2)^{1/3}$, and the Airy function is

$$A_i(y) = \int_{-\infty}^{+\infty} \frac{du}{2\pi} e^{\frac{i}{3}u^3 + iuy} \quad (10)$$

The Airy function satisfies the differential equation

$$A_i''(y) = yA_i(y), \quad A_i''(-ay) = -a^3yA_i(-ay) \quad (11)$$

The genus one case ($g=1$) has been studied in [7].

If one expands the hyperbolic sine function and the Airy functions in (9) up to relevant orders, we find a sum of six terms which, for $g = 2$, involve the following integrals:

$$\begin{aligned} I_1 &= \int_0^\infty dy y^5 A_i(y) A_i(-ay), \quad I_2 = \int_0^\infty dy y A_i''(y) A_i(-ay), \\ I_3 &= \int_0^\infty dy y A_i(y) A_i''(-ay), \quad I_4 = \int_0^\infty dy y A_i'(y) A_i'(-ay), \\ I_5 &= \int_0^\infty dy y^3 A_i'(y) A_i'(-ay), \quad I_6 = \int_0^\infty dy y^3 A_i(y) A_i'(-ay) \end{aligned} \quad (12)$$

A repeated use of (11) plus integrations by parts allows us to write all these integrals in terms of

$$A_i(0) = \frac{3^{-2/3}}{\Gamma(2/3)} = \frac{1}{2\pi 3^{1/3}} \Gamma\left(\frac{1}{3}\right), \quad A_i'(0) = -\frac{3^{-1/3}}{\Gamma(1/3)} = -\frac{1}{2\pi} \Gamma\left(\frac{2}{3}\right) \quad (13)$$

plus the integral

$$T = \int_0^\infty dy A_i(y) A_i'(-ay) \quad (14)$$

which cannot be reduced to $A_i(0)$ or $A_i'(0)$. For instance one finds

$$(1 + a^3)I_2 = A_i(0)^2 - 2T \quad (15)$$

and so on. However, all the T -dependence cancels when we sum up all the terms relevant to $g = 2$ in $U(s_1, s_2)$. For instance the sum of all terms of order $s_2^{\frac{16}{3}}$ is given by

$$\begin{aligned} &\frac{1}{5!} \frac{1}{16} 3^{4/3} (1 + a^3)^4 a^5 s_2^{\frac{16}{3}} I_1 + \frac{1}{2} \left(\frac{1}{4 \cdot 3^{1/3}} \right)^2 a^{17} s_2^{\frac{16}{3}} I_2 \\ &+ \frac{1}{2} \left(\frac{1}{4 \cdot 3^{1/3}} \right)^2 a^{-1} s_2^{\frac{16}{3}} I_3 - \left(\frac{1}{4 \cdot 3^{1/3}} \right)^2 a^8 s_2^{\frac{16}{3}} I_4 \\ &- \frac{1}{3!} \frac{1}{16} 3^{1/3} (1 + a^3)^2 a^{11} s_2^{\frac{16}{3}} I_5 + \frac{1}{3!} \frac{1}{16} 3^{1/3} a^2 (1 + a^3)^2 s_2^{\frac{16}{3}} I_6 \end{aligned} \quad (16)$$

and we add up the six terms and expand in powers of s_1 to the relevant orders we find

$$U(s_1, s_2)|_{g=2} = \frac{(A_i(0))^2}{32 \cdot 3^{2/3}} \left(-s_1^{14/3} s_2^{2/3} - \frac{11}{5} s_1^{11/3} s_2^{5/3} - \frac{17}{5} s_1^{8/3} s_2^{8/3} - \frac{11}{5} s_1^{5/3} s_2^{11/3} - s_1^{2/3} s_2^{14/3} \right). \quad (17)$$

From these results, we obtain the intersection numbers

$$\begin{aligned} \langle \tau_{0,1} \tau_{4,1} \rangle_{g=2} &= \frac{1}{864} \\ \langle \tau_{1,1} \tau_{3,1} \rangle_{g=2} &= \frac{11}{4320} \\ \langle \tau_{2,1} \tau_{2,1} \rangle_{g=2} &= \frac{17}{4320} \end{aligned} \quad (18)$$

Rather than computing the exact dependence in a of the terms proportional to $s_2^{16/3}$ and then re-expand in a to obtain the various terms of (17), we may proceed in a simpler way by expanding $A_i(-ay)$, $A'_i(-ay)$, $A''_i(-ay)$ for small a :

$$A_i(-ay) = A_i(0) - ay A'_i(0) + \frac{a^2}{2} y^2 A''_i(0) + \dots \quad (19)$$

and we then recover (18).

In the genus- three case ($g=3$), we have again ten distinct integrals $J_1 - J_{10}$ for the terms of order $s_2^8 a^m$, in the small s_1, s_2 expansion of (9).

$$\begin{aligned} J_1 &= \int_0^\infty dy y^7 A_i(y) A_i(-ay), \quad J_2 = \int_0^\infty dy y^5 A'_i(y) A_i(-ay) \\ J_3 &= \int_0^\infty dy y^5 A_i(y) A'_i(-ay), \quad J_4 = \int_0^\infty dy y^3 A'_i(y) A'_i(-ay) \\ J_5 &= \int_0^\infty dy y^3 A''_i(y) A_i(-ay), \quad J_6 = \int_0^\infty dy y^3 A_i(y) A''_i(-ay) \\ J_7 &= \int_0^\infty dy y A'''_i(y) A_i(-ay), \quad J_8 = \int_0^\infty dy y A_i(y) A'''_i(-ay) \\ J_9 &= \int_0^\infty dy y A''_i(y) A'_i(-ay), \quad J_{10} = \int_0^\infty dy y A'_i(y) A''_i(-ay) \end{aligned} \quad (20)$$

The genus 3 contribution for $U(s_1, s_2)$ is then expressed as the sum of four terms, $U^{(1)} - U^{(4)}$. The term $U^{(1)}$, which is related to J_1 , is

$$\begin{aligned} U^{(1)} &= \frac{9}{7! \cdot 64} s_1^{7/3} s_2^{17/3} (1 + a^3)^6 J_1 \\ &= \frac{3}{8960} s_2^8 (-15a^7 - 42a^8 + 90a^{10} + 63a^{11} \\ &\quad + 63a^{13} + 90a^{14} - 42a^{16} - 15a^{17}) A_i(0) A'_i(0). \end{aligned} \quad (21)$$

The term $U^{(2)}$, which is related to J_2 , is

$$\begin{aligned} U^{(2)} &= -\frac{1}{2560}s_2^8a^{13}(1+a^3)^4J_2 \\ &= -\frac{1}{2560}s_2^8a^{13}(30+72a-120a^3-90a^4+12a^6)A_i(0)A'_i(0). \end{aligned} \quad (22)$$

The term $U^{(3)}$, which is related to J_3 , is

$$\begin{aligned} U^{(3)} &= \frac{1}{2560}s_2^8a^4(1+a^3)^4J_3 \\ &= \frac{1}{2560}s_2^8a^5(-12+90a^2+120a^3-72a^5-30a^6)A_i(0)A'_i(0). \end{aligned} \quad (23)$$

The term $U^{(4)}$ from the sum of the contributions of J_4 to J_{10} . We have

$$\begin{aligned} J_{10} &= \frac{1}{1+a^3}(2a^3K_1-a^3K_2) \\ J_9 &= -K_2-J_{10} \\ J_8 &= \frac{a^3+2a^4-2a^6-a^7}{(1+a^3)^2}L \\ J_7 &= \frac{1}{a^3}J_8 \\ J_6 &= \frac{6a^3}{1+a^3}J_7 \\ J_5 &= -\frac{1}{a^3}J_6 \\ J_4 &= a^3J_5-3J_9 \end{aligned} \quad (24)$$

with $L = A_i(0)A'_i(0)$, and K_1, K_2 are given below. We have also the following relations between J_1, J_2 and J_3 ,

$$\begin{aligned} J_1 &= \frac{1}{1+a^3}(30J_5+12J_3) \\ J_2 &= \frac{1}{1+a^3}(-5J_5+4J_4) \\ J_3 &= \frac{1}{1+a^3}(-5a^3J_5-4J_4) \end{aligned} \quad (25)$$

Thus $U^{(4)}$ becomes

$$\begin{aligned} U^{(4)} &= \frac{1}{1152}s_2^8(a+5a^4-20a^7+23a^{10}+16a^{13} \\ &\quad -19a^{16}+13a^{19}+2a^{22}+(2a^2+13a^5-19a^8 \\ &\quad +16a^{11}+23a^{14}-20a^{17}+5a^{20}+a^{23}))A_i(0)A'_i(0). \end{aligned} \quad (26)$$

Since $a = (s_1/s_2)^{1/3}$, the above expression for $U(s_1, s_2)$ is a symmetric function in s_1 and s_2 . Denoting

$$s^{m+\frac{1+j}{3}} = t_{m,j} \quad (27)$$

and dividing $U(s_1, s_2)$ by $1/g^p$, i.e. $1/27$ in this case, we obtain the intersection numbers $\langle \tau_{m_1,j_1} \tau_{m_2,j_2} \rangle_{g=3}$ as the coefficient of $t_{m_1,j_1} t_{m_2,j_2}$ in $U(s_1, s_2)$ taking into account the spin factors. The following spin factor appears as a over all factor in $U(s_1, s_2)$ at genus 3.

$$A_i(0)A'_i(0) = -\frac{1}{(2\pi)^2 3^{1/3}} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) \quad (28)$$

where $\Gamma(\frac{1}{3})$, $\Gamma(\frac{2}{3})$ are spin 1 and spin 2 factors, respectively, as (7). All the integrals J_1, \dots, J_{10} are expressed by (28), and there are no terms like (14), which appeared in the integrals for the $g=1, g=2$ cases. Finally we have to compute the following terms

$$\begin{aligned} K_1 &= \int dy A''_i(y) A_i(-ay) = -A_i(0)A'_i(0) - K_2 \\ K_2 &= \int dy A'_i(y) A'_i(-ay). \end{aligned} \quad (29)$$

For these integrals, we find

$$K_1 = -\frac{1+a}{1+a^3} A_i(0)A'_i(0), \quad K_2 = \frac{a-a^3}{1+a^3} A_i(0)A'_i(0) \quad (30)$$

and all the integrals reduces to the spin factor (28). Summing up the results of $U(1)$ to $U(4)$, we obtain the intersection numbers for $p=3, g=3$,

$$\begin{aligned} \langle \tau_{0,0} \tau_{7,1} \rangle_{g=3} &= \frac{1}{31104}, & \langle \tau_{0,1} \tau_{7,0} \rangle_{g=3} &= \frac{1}{15552} \\ \langle \tau_{1,0} \tau_{6,1} \rangle_{g=3} &= \frac{5}{31104}, & \langle \tau_{1,1} \tau_{6,0} \rangle_{g=3} &= \frac{19}{77760} \\ \langle \tau_{2,0} \tau_{5,1} \rangle_{g=3} &= \frac{103}{217728}, & \langle \tau_{2,1} \tau_{5,0} \rangle_{g=3} &= \frac{47}{77760} \\ \langle \tau_{3,0} \tau_{4,1} \rangle_{g=3} &= \frac{443}{544320}, & \langle \tau_{3,1} \tau_{4,0} \rangle_{g=3} &= \frac{67}{77760} \end{aligned} \quad (31)$$

The above results are in complete agreement with the previous results [13, 15], which were obtained by recursion relations.

4 Intersection numbers for $p > 3$

For higher multicritical points the algebra is similar, except that we have to deal with generalized Airy functions. For instance for $p=4$ instead of $A_i(x)$

we have to work with $\phi(x)$ defined as

$$\phi(x) = \int_0^\infty dv e^{-\frac{1}{4}v^4 + vx} \quad (32)$$

which satisfies

$$\phi'''(x) = x\phi. \quad (33)$$

Then, similarly

$$U(s_1, s_2) = \frac{2}{(s_1 + s_2)(4s_2)^{1/4}} \int_0^\infty dx \int_0^\infty dv_1 dv_2 \text{sh}\left(\frac{s_1 + s_2}{2}(4s_1)^{1/4}x\right) \\ e^{-\frac{s_1^3}{2}(\frac{1}{4s_1})^{1/2}v_1^2 - \frac{s_2^3}{2}(\frac{1}{4s_2})^{1/2}v_2^2} e^{-\frac{1}{4}v_1^4 + xv_1 - \frac{1}{4}v_2^4 - axv_2} \quad (34)$$

where $a = (s_1/s_2)^{1/4}$. In complete analogy with the $p = 3$ case, a repeated use of integration by parts and of (33) leads to the expansion of $U(s_1, s_2)$. In the genus one case,

$$U(s_1, s_2)|_{g=1} = \frac{1}{4}(\phi''(0))^2 s_1^{1/4} s_2^{1/4} (s_1^2 + s_1 s_2 + s_2^2) \\ + \frac{1}{12}(s_1 s_2)^{3/4} (s_1 + s_2)(\phi(0))^2 \quad (35)$$

with

$$\phi''(0) = 2^{1/2}\Gamma(\frac{3}{4}), \quad \phi(0) = 2^{-1/2}\Gamma(\frac{1}{4}) \quad (36)$$

which provide the $j = 0, j = 2$ spin factors respectively. Replacing s_1, s_2 by t_m, j , ($s^{m+(1+j)/p} = t_{m,j}$),

$$U(s_1, s_2)|_{g=1} = \frac{1}{2}(t_{2,0}t_{0,0} + t_{1,0}t_{1,0} + t_{0,0}t_{2,0})(\Gamma(\frac{3}{4}))^2 \\ + \frac{1}{24}(t_{1,2}t_{0,2} + t_{0,2}t_{1,2})(\Gamma(\frac{1}{4}))^2 \quad (37)$$

Multiplying by a factor $\frac{1}{p^g}$, we obtain the intersection numbers as coefficients of (37) for $p = 4$ in the genus one case,

$$< \tau_{0,0}\tau_{2,0} >_{g=1} = \frac{1}{8}, \quad < \tau_{1,0}\tau_{1,0} >_{g=1} = \frac{1}{8}, \quad < \tau_{0,2}\tau_{1,2} >_{g=1} = \frac{1}{96} \quad (38)$$

For $g = 2, p = 4$, from the term $s_2^{\frac{18}{4}} s_1^{\frac{2}{4}}$, we have similarly

$$< \tau_{0,1}\tau_{4,1} >_{g=2} = \frac{1}{320} \quad (39)$$

For general p we have to deal with the generalized Airy functions $\phi(x)$ for $p > 2$, which satisfy the differential equation,

$$\phi^{(p)}(x) = x\phi(x) \quad (40)$$

where $\phi^{(p)}(x)$ means the p -th derivative of $\phi(x)$. The generalized Airy function has an integral representation,

$$\phi(y) = \int_0^\infty du e^{-\frac{u^p}{p} + yu}. \quad (41)$$

As examples of what the method can provide we give a few results : for the case $p = 5$, we obtain

$$\begin{aligned} \langle \tau_{1,3}\tau_{0,2} \rangle_{g=1} &= \langle \tau_{1,2}\tau_{0,3} \rangle_{g=1} = \frac{1}{60} \\ \langle \tau_{1,0}\tau_{1,0} \rangle_{g=1} &= \langle \tau_{0,0}\tau_{2,0} \rangle_{g=1} = \frac{1}{6} \\ \langle \tau_{0,1}\tau_{4,1} \rangle_{g=2} &= \frac{7}{1200}. \end{aligned} \quad (42)$$

For the case $p = 6$,

$$\langle \tau_{0,3}\tau_{1,3} \rangle_{g=1} = \frac{1}{36}, \quad \langle \tau_{0,2}\tau_{1,4} \rangle_{g=1} = \frac{1}{48}, \quad \langle \tau_{0,4}\tau_{1,2} \rangle_{g=1} = \frac{1}{48}. \quad (43)$$

For the case $p = 7$,

$$\begin{aligned} \langle \tau_{0,2}\tau_{1,5} \rangle_{g=1} &= \langle \tau_{1,2}\tau_{0,5} \rangle_{g=1} = \frac{1}{42} \\ \langle \tau_{0,4}\tau_{1,3} \rangle_{g=1} &= \langle \tau_{0,3}\tau_{1,4} \rangle_{g=1} = \frac{1}{28} \\ \langle \tau_{1,0}\tau_{1,0} \rangle_{g=1} &= \langle \tau_{0,0}\tau_{2,0} \rangle_{g=1} = \frac{1}{4}. \end{aligned} \quad (44)$$

5 The p dependence of the intersection numbers

In a previous article [18], we have considered the intersection numbers with one marked point for arbitrary p , and found results such as

$$\begin{aligned} \langle \tau_{1,0} \rangle_{g=1} &= \frac{p-1}{24} \\ \langle \tau_{n,j} \rangle_{g=2} &= \frac{(p-1)(p-3)(1+2p)}{p \cdot 5! \cdot 4^2 \cdot 3} \frac{\Gamma(1 - \frac{3}{p})}{\Gamma(1 - \frac{1+j}{p})} \\ \langle \tau_{n,j} \rangle_{g=3} &= \frac{(p-5)(p-1)(1+2p)(8p^2 - 13p - 13)}{p^2 \cdot 7! \cdot 4^3 \cdot 3^2} \frac{\Gamma(1 - \frac{5}{p})}{\Gamma(1 - \frac{1+j}{p})} \\ \langle \tau_{n,j} \rangle_{g=4} &= \frac{(p-7)(p-1)(1+2p)(72p^4 - 298p^3 - 17p^2 + 562p + 281)}{p^3 \cdot 9! \cdot 4^4 \cdot 15} \\ &\quad \times \frac{\Gamma(1 - \frac{7}{p})}{\Gamma(1 - \frac{1+j}{p})} \end{aligned} \quad (45)$$

with $n = 2g - 1 + \frac{2g-2-j}{p}$. In the large p limit, the intersection numbers $\langle \tau_{n,j} \rangle_g$ behave as

$$\langle \tau_{n,j} \rangle_g = \frac{B_g}{(2g)!(2g)} p^g + O(p^{g-1}) \quad (46)$$

with B_g is a Bernoulli number, $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}$. Note the well known relation to $\zeta(2g)$ as

$$\frac{B_g}{(2g)!(2g)} = \frac{1}{(2\pi)^{2g}g} \zeta(2g) \quad (47)$$

We have derived (46) from $U(s)$ in the large p limit. The one point function $U(s)$ has the following expression[16],

$$U(s) = \frac{1}{Ns} \int \frac{du}{2i\pi} \exp\left(-\frac{c}{p+1} \left((u + \frac{1}{2}s)^{p+1} - (u - \frac{1}{2}s)^{p+1}\right)\right) \quad (48)$$

With $s = \frac{\sigma}{p}$, and $u^{p+1} = x^2$, we have

$$U(s) = \frac{2}{N\sigma} \int \frac{dx}{2i\pi} x^{-1+\frac{2}{p}} e^{-\frac{c}{p+1} x^2 (e^{\sigma/2} - e^{-\sigma/2})} \quad (49)$$

Thus we obtain

$$U(s) = \frac{2}{N\sigma} \Gamma\left(\frac{2}{p}\right) \left(\frac{2c}{p+1} \text{sh}\frac{\sigma}{2}\right)^{-1/p} \quad (50)$$

This may be written as

$$U(s) = \frac{2}{N\sigma} \Gamma\left(\frac{2}{p}\right) \left(\frac{2c}{p+1}\right)^{-\frac{1}{2}} \left(\frac{\sigma}{2}\right)^{-\frac{1}{p}} \exp\left(-\frac{1}{p} \log\frac{\text{sh}\frac{\sigma}{2}}{\frac{\sigma}{2}}\right) \quad (51)$$

and expanding the exponent in $\frac{1}{p}$, we find

$$U(s) = \frac{2}{N\sigma} \Gamma\left(\frac{2}{p}\right) \left(\frac{2c}{p+1}\right)^{-\frac{1}{2}} \left(\frac{\sigma}{2}\right)^{-\frac{1}{p}} \left(1 - \frac{1}{p} \log\left(\frac{\text{sh}(\frac{\sigma}{2})}{(\frac{\sigma}{2})}\right)\right) \quad (52)$$

If we use the expansion

$$\log\left(\frac{\text{sh}\frac{\sigma}{2}}{\frac{\sigma}{2}}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n \sigma^{2n}}{(2n)! 2n} \quad (53)$$

and drop the factors $\left(\frac{2c}{p+1}\right)^{-\frac{1}{2}}$, and $(\sigma/2)^{-1/p}$ which are close to one in the large p limit, we obtain

$$U(s) = \left(1 - \frac{1}{p} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{(2n)! 2n} \sigma^{2n}\right) \frac{p}{N\sigma} \Gamma\left(1 + \frac{2}{p}\right) \quad (54)$$

Since the intersection numbers $\langle \tau_{n,j} \rangle_g$ are related to $U(s)$ by [16]

$$U(s) = \sum_g \langle \tau_{n,j} \rangle_g \frac{1}{N\pi} \Gamma(1 - \frac{1+j}{p}) s^{(2g-1)(1+\frac{1}{p})} p^{g-1} \quad (55)$$

with $(p+1)(2g-1) = pn + j + 1$, we have rederived the large p behavior of (46).

From (53), taking a derivative with respect to σ , gives,

$$\frac{1}{e^\sigma - 1} + \frac{1}{2} - \frac{1}{\sigma} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{(2n)!} \sigma^{2n-1}, \quad (56)$$

Using this relation one obtains

$$\frac{d}{d\sigma}(\sigma U(\sigma)) = \frac{1}{\sigma} - \frac{1}{2} - \frac{1}{e^\sigma - 1} \quad (57)$$

The di-gamma function $\psi(z)$ has the following expression,

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \log z - \frac{1}{2z} - \int_0^\infty d\sigma \left(\frac{1}{2} - \frac{1}{\sigma} + \frac{1}{e^\sigma - 1} \right) e^{-\sigma z}. \quad (58)$$

From (57) and (58) we find thus in the large p limit,

$$\begin{aligned} \frac{d}{dz} \log \Gamma(z) &= \log z - \frac{1}{2z} + \int_0^\infty d\sigma \left(\frac{d}{d\sigma}(\sigma U(\sigma)) \right) e^{-\sigma z} \\ &= \log z - \frac{1}{2z} - z \frac{d}{dz} \int d\sigma U(\sigma) e^{-\sigma z} \end{aligned} \quad (59)$$

The last integral is related to the density of states. In (2), s is replaced by $s = it$, and if we replace z by iE , and take the imaginary part, we obtain the density of states $\rho(E)$. After integration by parts, we obtain

$$\rho(E) = \frac{d}{dE} \text{Im} \log \Gamma(iE) - \frac{\pi}{2} - \frac{1}{2E} \quad (60)$$

We will discuss this expression in connection to the density of states of the $SL(2, R)/U(1)$ black hole sigma model in the next section.

Next we consider the two point correlation function $U(s_1, s_2)$. For general p , $U(s_1, s_2)$ is expressed as

$$\begin{aligned} U(s_1, s_2) &= \frac{2}{(s_1 + s_2)(ps_2)^{1/p}} \int_0^\infty dx \int_0^\infty dv_1 dv_2 \text{sh} \left(\frac{s_1 + s_2}{2} (ps_1)^{1/p} \right) \\ &\quad e^{-\frac{v_1^p}{p} + xv_1 - \frac{p(p-1)}{24} s_1^3 (ps_1)^{\frac{2-p}{p}} v_1^{p-2} + \dots} e^{-\frac{v_2^p}{p} - axv_2 - \frac{p(p-1)}{24} s_2^3 (ps_2)^{\frac{2-p}{p}} v_2^{p-2} + \dots} \end{aligned} \quad (61)$$

The exponent of (61) follows from the binomial expansion,

$$(u + \frac{s}{2})^{p+1} = u^{p+1} + (p+1)u^p(\frac{s}{2}) + (p+1)p\frac{1}{2}u^{p-1}(\frac{s}{2})^2 + \dots \quad (62)$$

and we use $c(p+1) = 1$, $u^p s = v^p/p$. As in the case of $p=3$, polynomials in a (21) give the intersection numbers. Therefore we expand (61) in power series of a . At lowest order in a , we obtain two terms from (61),

$$\begin{aligned} U_1 &= \frac{1}{3!4}(s_1 + s_2)^2 \frac{(ps_1)^{\frac{3}{p}}}{(ps_2)^{\frac{1}{p}}} \int dx x^3 \phi(x) \phi(-ax) \\ U_2 &= -\frac{p(p-1)}{24} (\frac{s_1}{s_2})^{\frac{1}{p}} s_2^3 (ps_2)^{\frac{2-p}{p}} \int dx x \phi(x) \phi^{(p-2)}(-ax) (-a)^{2-p} \end{aligned} \quad (63)$$

From U_2 we find a term proportional to $as_2^{2+\frac{2}{p}}$, namely

$$\Delta U_2 = \frac{p-1}{24} p^{\frac{2}{p}} a s_2^{2+\frac{2}{p}} (\phi^{(p-2)}(0))^2 \quad (64)$$

with

$$\begin{aligned} \phi^{(p-2)}(0) &= \int_0^\infty du u^{p-2} e^{-\frac{u^p}{p}} \\ &= p^{-\frac{1}{p}} \Gamma(1 - \frac{1}{p}). \end{aligned} \quad (65)$$

Since $s_2^{2+\frac{1}{p}} s_1^{\frac{1}{p}} = t_{2,0} t_{0,0}$, we obtain

$$< \tau_{0,0} \tau_{2,0} >_{g=1} = \frac{p-1}{24} \quad (66)$$

From U_1 and U_2 , we collect terms proportional to $a^3 s_2^{2+\frac{2}{p}}$ and obtain

$$< \tau_{0,2} \tau_{1,p-2} >_{g=1} = \frac{p-3}{24p} \quad (67)$$

This result agrees with those obtained previously for $p = 4, 5, 6$ and 7 in (38), (42), (43) and (44). The intersection number (67) can be neglected in the large p limit in comparison with (66).

Similarly one obtains the $g=2$ terms from the coefficients of $a^m s_2^{4+\frac{4}{p}}$ ($m=1,2,3$),

$$\begin{aligned} < \tau_{0,0} \tau_{4,2} >_{g=2} &= \frac{(p-1)(p-3)(2p+1)}{5760p} \\ < \tau_{0,1} \tau_{4,1} >_{g=2} &= \frac{(p-1)(p-2)(p+2)}{2880p} \\ < \tau_{0,2} \tau_{4,0} >_{g=2} &= \frac{(p-1)(p-3)(2p+11)}{5760p} \end{aligned} \quad (68)$$

For the particular values of $p = 3, 4, 5$, the above expressions agree with the previous results (18), (39) and (42) for the genus two case.

From the $a^5 s_2^{4+\frac{4}{p}}$ term, one finds

$$< \tau_{0,4} \tau_{3,p-2} >_{g=2} = \frac{2p^3 + 13p^2 - 158p + 215}{5760p^2} \quad (69)$$

which is valid for $p \geq 6$.

In the large p limit, the three terms of (68) become equal, and coincide with the result for the one point intersection number (46).

$$< \tau_{0,m} \tau_{4,2-m} >_{g=2} = \frac{B_2 p^2}{4! \cdot 4} \quad (p \rightarrow \infty) \quad (70)$$

Note that (69) is order p , and is negligible compared to (68).

From the terms $a^m s_2^{6+\frac{6}{p}}$ in the small a expansion of $U(s_1, s_2)$, we obtain the $g=3$ (genus 3) terms. In the case $m=1$, we have

$$< \tau_{0,0} \tau_{6,4} >_{g=3} = \frac{(p-1)(p-5)(2p+1)(8p^2 - 13p - 13)}{p^2 \cdot 7! 4^3 3^2} \quad (p > 5) \quad (71)$$

This is identical to $< \tau_{5,4} >_{g=3}$ in (45). The identity follows from the string equation, in which the insertion of $\tau_{0,0}$ reduces the intersection number from s to $s-1$ marked points:

$$< \tau_{0,0} \prod_{i=1}^s \tau_{n_i, j_i} >_g = \sum_{l=1}^s < \tau_{n_l-1, j_l} \prod_{i=1, i \neq l}^s \tau_{n_i, j_i} >_g \quad (72)$$

In our formulation, this string equation follows from the integral representation for the intersection numbers, when one collects the terms proportional to a . By explicit calculation of two marked points, we verified this string equation. It might be possible to verify this string equation for n -marked points by the taking account of the term of a .

From $a^2 s_2^{6+\frac{6}{p}}$, we have for $p > 5$,

$$< \tau_{0,1} \tau_{6,3} >_{g=3} = \frac{(p-1)(p-2)(p-4)(p+2)(2p+1)}{p^2 \cdot 7! \cdot 8 \cdot 3^2} \quad (73)$$

From $a^3 s_2^{6+\frac{6}{p}}$,

$$< \tau_{0,2} \tau_{6,2} >_{g=3} = \frac{(p-1)(p-3)(16p^3 + 34p^2 - 155p - 129)}{p^2 \cdot 7! \cdot 64 \cdot 3^2} \quad (74)$$

In the large p limit, these $g=3$ terms exhibit same behavior as in (46),

$$< \tau_{0,m} \tau_{6,4-m} >_{g=3} = \frac{B_3}{6! \cdot 6} p^3 + O(p^2) \quad (p \rightarrow \infty) \quad (75)$$

6 Analytic continuation to negative p

One may analytically continue the integral representations of the correlation functions to negative values of p . This continuation was already examined in [18], and we recall some of the results here :

$$U(s) = \frac{1}{Ns} \int \frac{du}{2i\pi} e^{-c[(u+\frac{1}{2}s)^{p+1} - (u-\frac{1}{2}s)^{p+1}]} \quad (76)$$

where $c = \frac{N}{p^2-1} \sum \frac{1}{a_\alpha^{p+1}}$.

Expanding the exponent, we obtain

$$U(s) = \int \frac{du}{2i\pi} \exp[-c(su^p + \frac{p(p-1)}{3!4} s^3 u^{p-2} + \frac{p(p-1)(p-2)(p-3)}{5!4^2} s^5 u^{p-4} + \dots)]. \quad (77)$$

This integrals yield Gamma functions after the replacement $u = (\frac{t}{cs})^{1/p}$,

$$\begin{aligned} U(s) &= \frac{1}{Nsp} \cdot \frac{1}{(cs)^{1/p}} \int_0^\infty dt t^{\frac{1}{p}-1} e^{-(t + \frac{p(p-1)}{3!4} s^{2+\frac{2}{p}} c^{\frac{1}{p}} t^{1-\frac{2}{p}} + \frac{p(p-1)(p-2)(p-3)}{5!4^2} s^{4+\frac{4}{p}} c^{\frac{4}{p}} t^{1-\frac{4}{p}} + \dots)} \\ &= \frac{1}{Nsp} \cdot \frac{1}{(cs)^{1/p}} \left[-\frac{p-1}{24} c^{\frac{2}{p}} y \Gamma(1 - \frac{1}{p}) + \frac{(p-1)(p-3)(1+2p)}{5! \cdot 4^2 \cdot 3} y^2 \Gamma(1 - \frac{3}{p}) \right. \\ &\quad - \frac{(p-5)(p-1)(1+2p)(8p^2 - 13p - 13)}{7!4^3 3^2} y^3 \Gamma(1 - \frac{5}{p}) \\ &\quad + (p-7)(p-1)(1+2p)(72p^4 - 298p^3 - 17p^2 + 562p + 281) \\ &\quad \times \left. \frac{1}{9!4^4 15} y^4 \Gamma(1 - \frac{7}{p}) \dots \right] \quad (78) \end{aligned}$$

with $y = c^{\frac{2}{p}} s^{2+\frac{2}{p}}$.

From this expansion, we obtain the intersection numbers for one marked point as (45). The intersection number $\langle \tau_{n,j} \rangle_g$ is obtained from the term $y^g \Gamma(1 - \frac{1}{p} - \frac{j}{p})$ in (78).

The continuation to $p < 0$ is straightforward. The t -integral in (78) can be changed to v by $t = \frac{1}{v}$, ($0 < v < \infty$), and one obtains the small s expansion for negative p . Therefore the expression for the intersection numbers (45) can be analytically continued to negative p . This analytic continuation can also be done for two marked points, since we have computed them in the previous sections for general p . For instance, from (45), we have the intersection numbers for $p = -3$,

$$\begin{aligned} \langle \tau_{1,0} \rangle_{g=1} &= -\frac{1}{6}, \quad \langle \tau_{3,2} \rangle_{g=2} = \frac{1}{144} \\ \langle \tau_{6,1} \rangle_{g=3} &= -\frac{35}{34992} \end{aligned} \quad (79)$$

In a previous article [18], we have computed the intersection numbers $< \tau_{1,0} >_g$ for the case of $p = -1$ from $U(s)$, which provides the orbifold Euler characteristics $\chi(\mathcal{M}_{g,1})$ with one marked point,

$$< \tau_{1,0} >_g = \chi(\mathcal{M}_{g,1}) = \zeta(1 - 2g) = -\frac{B_g}{2g} \quad (80)$$

with the Bernoulli number B_g , ($B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \dots$). The s-point orbifold Euler characteristics $\chi(\mathcal{M}_{g,s})$ may be obtained from the dilaton equation:

$$< \tau_{1,0} \tau_{n_1, j_1} \cdots \tau_{n_k, j_k} >_g = (2g - 2 + k) < \tau_{n_1, j_1} \cdots \tau_{n_k, j_k} >_g \quad (81)$$

Since the Euler characteristics with s marked points is $< \tau_{1,0} \cdots \tau_{1,0} >_g$, the dilaton equation yields from (80),

$$\chi(\mathcal{M}_{g,s}) = < (\tau_{1,0})^s >_g = -\frac{2g-1}{(2g)!} (2g+s-3)! B_g \quad (82)$$

This agrees with previous results obtained in [20, 21, 22].

For $p = -2$, we have considered previously the equivalence with the unitary matrix model in a matrix source [26].

The central charge of the gauged Wess-Zumino-Witten model with symmetry $SU(2)_k/U(1)$ is

$$C = 2 - \frac{6}{k+2} \quad (83)$$

Changing p to $p = -p'$, k to $k = -k'$ ($p < 0, k < 0$), we have $p' = k' - 2$, and the central charge C is given by

$$C = 2 + \frac{6}{k' - 2} \quad (84)$$

The analytic continuation to negative p yields a gauged WZW model for $SL(2, R)_{k'}/U(1)$. It is known that this model represents a black hole σ model[9], in particular for the value $k' = \frac{9}{4}$ ($p = -\frac{1}{4}$), for which the central charge C becomes 26.

The density of states for the $SL(2, R)/U(1)$ black hole has been studied in [24, 25, 27],

$$\rho(E) = \frac{1}{\pi} \log \epsilon + \frac{1}{4\pi i} \frac{d}{dE} \log \frac{\Gamma(-iE + \frac{1}{2} - m) \Gamma(-iE + \frac{1}{2} + \tilde{m})}{\Gamma(+iE + \frac{1}{2} + \tilde{m}) \Gamma(+iE + \frac{1}{2} - m)} \quad (85)$$

in which ϵ is a regularization factor, and $m = \frac{1}{2}(n - kw)$, $\tilde{m} = -\frac{1}{2}(kw + n)$ are eigenvalues of J_0^3 and \bar{J}_0^3 in CFT ($J_0^3 - \bar{J}_0^3 = n$, $J_0^3 + \bar{J}_0^3 = -kw$). If we neglect m , \tilde{m} , and the $\frac{1}{2}$ terms in the large E limit, we obtain

$$\rho(E) = \frac{1}{\pi} \log \epsilon + \frac{1}{2\pi i} \frac{d}{dE} \log \frac{\Gamma(-iE)}{\Gamma(+iE)} \quad (86)$$

or

$$\rho(E) = \frac{2}{\pi} \frac{d}{dE} \text{Im} \log \Gamma(-iE) \quad (87)$$

This expression agrees with (60), obtained from the intersection numbers for large p . We have scaled $s = \sigma/p$, and the expression (60) is valid for small s . Therefore, the Fourier transform of $U(s)$ gives the large E behavior, in which the terms m, \tilde{m} and $1/2$ in (85) can be neglected.

7 Discussion

In this article, we have shown that the correlation functions $U(s_1, s_2, \dots, s_n)$ of a Gaussian matrix model in a tuned external source, provide the intersection numbers for p -spin curves. For instance, from the two point function $U(s_1, s_2)$, in the case of $p=3$, the intersection numbers are computed up to genus 3,

We have also computed the intersection numbers for general p . They are given by power series in a , $a = (\frac{s_1}{s_2})^{\frac{1}{p}}$. Then we have considered the large p behavior for the two point functions. The density of states $\rho(E)$ becomes a di-gamma function in the large p limit, and this expression agrees with the density of states of a $SL(2, R)_k/U(1)$ WZW model, which has been studied in the context of two dimensional black hole solutions. The n -point correlation functions $U(s_1, \dots, s_n)$ are known through the determinant of a kernel for the p -spin curve case. It will be interesting to investigate further the detailed comparison of those correlation functions, between $SL(2, R)_k/U(1)$ WZW theory and the intersection numbers for negative p -spin curves.

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